Math 215 – Midterm 2, November 9, 2023, 6-7:30 pm

Instructions

1. **Do not open this exam until you are told to do so.**
2. This exam contains 8 problems on 15 sheets of paper including the front cover. The second to last sheet is left blank for scratch work. The last sheet contains a list of formulas.
3. Do not separate pages of this exam, other than the formula sheet at the end of the exam. *Separated pages will not be graded.*
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem. Your answer should be clearly indicated, and your work should be clear. Include units in your answer where that is appropriate. A correct answer with no supporting work will receive only one point.
6. If needed, you may continue your solution on the back side of a page.
7. This is a closed book exam. Electronic devices, calculators, and note-cards are not allowed. **Turn off all cell phones, remove all headphones, and place any watch you are using on the desk in front of you.**

Name:

UMID:

Section:

The sections meet at the following times:

Section	10	30	50	60	70	80	90
Meeting time	8 am	9 am	11 am	12 pm	1 pm	2 pm	3 pm

Distribution of Marks

Question:	1	2	3	4	5	6	7	8	Total
Points:	12	12	15	12	12	14	12	12	101

This page will not be graded

Question 1:

For each statement below, indicate clearly whether the statement is true or false by circling the appropriate choice.

(a) (2 points) If $f(x,y) > g(x,y)$ for all (x,y) in some region R, then the surface area of the graph of f is larger than the surface area of the graph of g over R. TRUE FALSE

(b) (2 points) If a differentiable function $f(x,y)$ has a saddle point at (a,b), then there exists a unit vector $\hat{u}$ such that $D_{\hat{u}} f(a,b) < 0$. TRUE FALSE

(c) (2 points) $\left(\int_0^1 f(x)\,dx\right)^2 = \int_0^1 \int_0^1 \left(f(x)^2\right)\,dx\,dy$ TRUE FALSE

(d) (2 points) The vector field $\vec{F}(x,y) = \langle -y,\, x\rangle$ is a conservative vector field. TRUE FALSE

(e) (2 points) $\int_{-1}^1 \int_0^1 e^{x^2+y^2} \sin(x)\,dx\,dy = 0$ TRUE FALSE

(f) (2 points) Suppose that $\vec{F}$ and $\vec{G}$ are conservative vector fields on $\mathbb{R}^3$. Then the vector field $\vec{H} = \vec{F} + \vec{G}$ is also conservative. TRUE FALSE

Extra writing space for Question 1

Question 2:

Find the points on the ellipse $3x^2 + 4y^2 = 48$ closest to and furthest from the line $x + 2y = 10$. (It may help to draw a picture and clearly define the function you are trying to find the extremal values of, as well as any constraints that are given. I remind you all that there is a formula sheet at the end of the exam.)

Extra writing space for Question 2

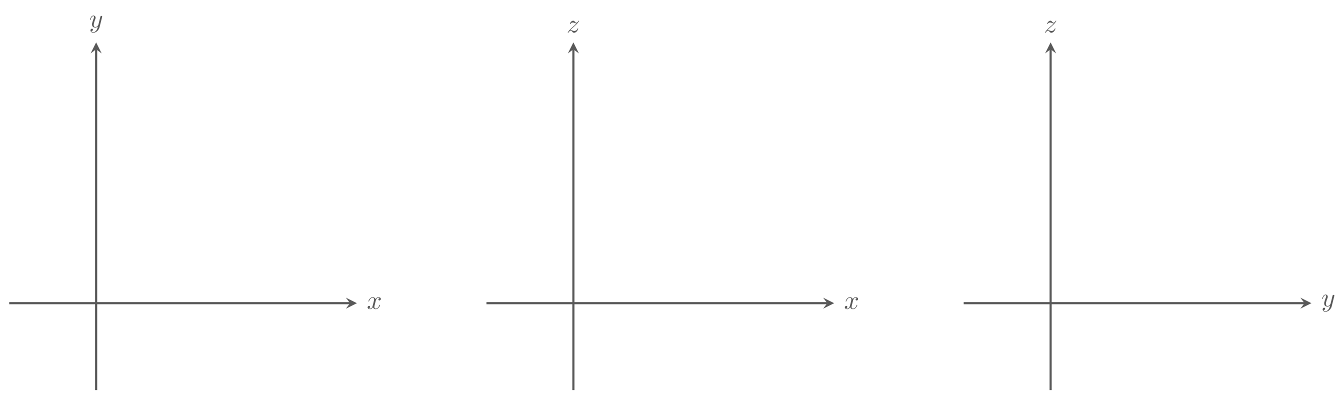
Question 3:

Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a continuously differentiable function everywhere, and consider the following integral:

$$\iiint_E f(x,y,z)\,dV = \int_0^1 \int_x^1 \int_0^{1-y^2} f(x,y,z)\,dz\,dy\,dx$$

(a) (3 points) Sketch the volume E.

(b) (6 points) Sketch the projection of E into the xy, xz, and yz planes (i.e. the shadow that E casts into those planes). Make sure to label the curves delineating the boundaries of the regions.



(c) (6 points) Rewrite the given integral with integration order $dx\,dy\,dz$.

Extra writing space for Question 3

Question 4:

Suppose that the density function of a rectangular lamina, $0 \leq x \leq 1$, $0 \leq y \leq \dfrac{\pi}{3}$, is given by

$$\rho(x, y) = \int_{y}^{\pi/3} \sin(t^2)\, dt$$

Find the mass of the lamina.

Extra writing space for Question 4

Question 5:

Find the surface area of the part of the cylinder $x^2 + z^2 = 4$ that lies over the region in the xy-plane bounded by $y = x$, $y = -2x$, and $x = 2$.

Extra writing space for Question 5

Question 6:

Let $f(x,y) = \left(x^2+y^2\right)e^{x^2+xy-y^2}$, and let R be the region in the first quadrant bounded by the curves $x^2-y^2=2$, $x^2-y^2=4$, $xy=\ln 2$, and $xy=\ln 4$.

(a) (2 points) Sketch the region of integration. Be sure to label the boundaries of R.

(b) (3 points) Find a coordinate transformation $(x,y)\mapsto(u,v)$ such that the region R (in the xy-plane) becomes a "nice" region D in the uv-plane, and sketch the region D. Your answer here should include formulas relating u, v, x, and y.

Extra writing space for Question 6

(c) (4 points) Recall the Jacobian of a transformation $(u, v) \mapsto (x, y)$ is given by $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$.

You may take as a given that $\dfrac{\partial(x,y)}{\partial(u,v)} = \left(\dfrac{\partial(u,v)}{\partial(x,y)}\right)^{-1}$.

Compute the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$.

Extra writing space for Question 6

(d) (5 points) Evaluate $\iint_R f(x, y)\, dA$.

Extra writing space for Question 6

Question 7:

Let E be the region inside the sphere of radius 4 centered at the origin for which $z \geq \sqrt{x^2 + y^2}$.

(a) (4 points) Set up, but do not evaluate, the integral $\iiint_E x\,dV$ in *spherical coordinates.*

Extra writing space for Question 7

(b) (4 points) Set up, but do not evaluate, the integral $\iiint_E x\, dV$ in *cylindrical coordinates*.

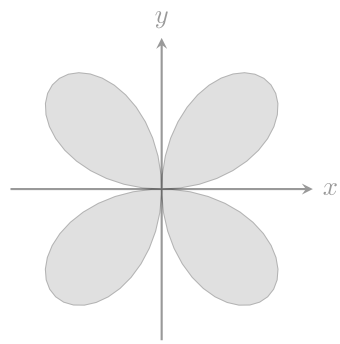
Extra writing space for Question 7

(c) (4 points) Suppose we added the additional restriction that $z \geq 1$ (in addition to the already established conditions). Would it be better to integrate $\iiint_E x\, dV$ in cylindrical or spherical coordinates? Explain your reasoning.

Extra writing space for Question 7

Question 8:

Let R be the four leaf clover curve defined by $r = \sin(2\theta)$, $0 \leq \theta < 2\pi$. Find the area enclosed by this curve (the shaded region in the diagram below).



Extra writing space for Question 8

Page for scratch work. Not graded if torn off.

Page for scratch work. Not graded if torn off.

Formula sheet. Not graded. May be torn off.

(Not all of these formulas are necessarily needed for this exam.)

- $\sin^2 x+\cos^2 x = 1$, $\cos(2x) = \cos^2 x-\sin^2 x$, $\sin(2x) = 2\sin x\cos x$, $\sin^2 x = \dfrac{1-\cos(2x)}{2}$, $\cos^2 x = \dfrac{1+\cos(2x)}{2}$
- $\sin\left(\dfrac{\pi}{2} - x\right) = \cos x$, $\cos\left(\dfrac{\pi}{2} - x\right) = \sin x$
-

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\cos x$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
$\sin x$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1

- $\dfrac{d}{dx}\sin x = \cos x$, $\dfrac{d}{dx}\cos x = -\sin x$
- $\mathrm{proj}_{\mathbf{u}}\mathbf{v} = \left(\dfrac{\mathbf{v}\cdot\mathbf{u}}{\|\mathbf{u}\|}\right)\dfrac{\mathbf{u}}{\|\mathbf{u}\|}$, $\mathrm{comp}_{\mathbf{u}}\mathbf{v} = \left(\dfrac{\mathbf{v}\cdot\mathbf{u}}{\|\mathbf{u}\|}\right)$
- $\langle a,\, b,\, c\rangle \times \langle d,\, e,\, f\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{vmatrix} = \langle bf-ce,\, cd-af,\, ae-bd\rangle$
- $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$, $\mathbf{a}\times(\alpha\mathbf{b}) = (\alpha\mathbf{a})\times\mathbf{b} = \alpha\,(\mathbf{a}\times\mathbf{b})$
- The area of the parallelogram determined by the vectors $\mathbf{v}_1 = \langle a,\, b,\, c\rangle$ and $\mathbf{v}_2 = \langle d,\, e,\, f\rangle$ is $\|\mathbf{v}_1\times\mathbf{v}_2\|$. The area of the triangle spanned by the same two vectors is one half of the area of the parallelogram.
- The volume of the parallelepiped determined by the three vectors $\mathbf{v}_1 = \langle a,\, b,\, c\rangle$, $\mathbf{v}_2 = \langle d,\, e,\, f\rangle$, and $\mathbf{v}_3 = \langle g,\, h,\, i\rangle$ is
$$|\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)| = \text{absolute value of } \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$
- The distance from a point (a,b,c) to a plane $Ax+By+Cz+D=0$ is $\dfrac{|Aa+Bb+Cc+D|}{\sqrt{A^2+B^2+C^2}}$.
- The circumference of a circle of radius a is $2\pi a$.
- The area of a disk of radius a is πa^2.
- The volume of a ball of radius a is $\dfrac{4\pi a^3}{3}$.
- The surface area of a sphere of radius a is $4\pi a^2$.
- The surface area of the graph of a function $f(x,y)$ over a region D is $\displaystyle\iint_D \sqrt{1+f_x^2+f_y^2}\,dA$.
- Common coordinate systems:

Cartesian	Cylindrical	Spherical
x	$r\cos\theta$	$\rho\cos\theta\sin\phi$
y	$r\sin\theta$	$\rho\sin\theta\sin\phi$
z	z	$\rho\cos\phi$
$dx\,dy\,dz$	$r\,dr\,d\theta\,dz$	$\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

This page will not be graded. May be torn off.